

Lattice Chirality *

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The external fermion propagator and the internal fermion propagator in the overlap are given by different matrices. A generic problem (formulated by Pelissetto) faced by all chiral, non-local, propagators of Rebbi type is avoided in this manner. Nussinov-Weingarten-Witten mass inequalities are exactly preserved. It is sketched how to obtain simple lattice chiral Yukawa models and simple expressions for covariant currents. Going beyond my oral presentation, I have added to the write-up several comments on Niedermayer's talk. His transparencies are available on the internet.

Chiral symmetry was built into the overlap from day one. Each Weyl fermion is represented by an infinite tower and coherent unitary rotations of left-towers into linear combinations of left towers (the same holds for right towers) are exact global symmetries in vector-like gauge theories. These are symmetries of the fermion quantum system underlying the overlap and are realized canonically. There are also global $U(1)$ symmetries associated with each tower. Fermion correlation functions are produced by taking matrix elements of strings of creation/annihilation operators between a reference state and a fermionic ground state. The ground state depends parametrically on the gauge background, but the reference state can be chosen not to. The ground state can transform nontrivially under one linear combination of the $U(1)$ s, axial- $U(1)$, but is a singlet under all other global chiral symmetries as long as the gauge background is smooth enough. This provides a lattice realization of 't Hooft's solution to the $U(1)$ -problem. The backgrounds where axial- $U(1)$ is violated contain net topological charge. In addition, a certain choice for the external fermion propagator produces an exact lattice realization of the Nussinov-Weingarten-Witten mass inequalities. All of the above is fully explained in section 9 of [1]. In other sections evidence is provided for correct realization of chiral anomalies. Thus, spontaneous chiral symmetry breakdown is not only plausible beyond reasonable doubt, but potentially rigorously provable.

The simplified formula obtained for the overlap in [2] is an expression of the matrix element of unity, the chiral determinant, $\det(D)$ with $D = (1 + V)/2$. Here, $V = \gamma_5 \epsilon(H)$, with $H = \gamma_5 D_W$, where D_W is the Wilson-Dirac operator with hopping larger than critical. Let the size of D_W be $\mathcal{N} \times \mathcal{N}$. D is obtained by starting from a $2\mathcal{N} \times 2\mathcal{N}$ hamiltonian $H'_2 = \begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix}$.

By conjugation with $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ one gets $H_2 = \begin{pmatrix} 0 & H \\ H & 0 \end{pmatrix}$. To get the external fermion propagator one needs the projectors on the positive and negative eigenspaces of H_2 (see section 5 of [1]). In backgrounds carrying zero lattice topological charge they are $2\mathcal{N} \times \mathcal{N}$ rectangular matrices, $P_{\pm}(H) = \frac{1}{\sqrt{2}} (\epsilon(H) \pm 1)$. The reference state is obtained by substituting γ_5 for H (One can easily work with a gauge field dependent reference state, substituting H_0 for H , where H_0 differs from H in that its hopping parameter is below the critical value.) Since $\epsilon(\gamma_5) = \gamma_5$ we have $P_{\pm}(\gamma_5) = \frac{1}{\sqrt{2}} (\gamma_5 \pm 1)$.

Equations (5.7) and (5.19) in [1] provide an expression for the propagator \tilde{G}_2 :

$$\tilde{G}_2 = P_+^\dagger(H) \frac{1}{P_+(\gamma_5)P_+^\dagger(H)} P_+(\gamma_5) = \frac{1}{1+V} \begin{pmatrix} 1 & \gamma_5 \\ \gamma_5 & 1 \end{pmatrix} \quad (1)$$

Rotating back to the original frame, we get

$$\tilde{G}'_2 = \frac{1}{1+V} \begin{pmatrix} 1+\gamma_5 & 0 \\ 0 & 1-\gamma_5 \end{pmatrix} \quad (2)$$

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As noted in [1] we see that only half the modes propagate, so we have effectively an $\mathcal{N} \times \mathcal{N}$ restricted propagator. It is chiral, so has only left-left and right-right entries in the chiral basis: $\tilde{g}_{RR} = \tilde{g}_{LL} = \frac{2}{1+V}$. (The chiral projectors $\frac{1 \pm \gamma_5}{2}$ are assumed implied by the subscripts.) However, it violates the continuum relation $g_{RR} = -g_{LL}^\dagger$. To restore this essential relation for NWW mass inequalities the ordering ambiguity inherent in the overlap construction of fermionic correlation functions was exploited in [1] to obtain another expression, given in equation (5.22):

$$g_{RR} = -g_{LL}^\dagger = \frac{2}{1+V} - 1 = \frac{1-V}{1+V} \quad (3)$$

These can be assembled into a chiral, non-local propagator of Rebbi type:

$$g = \frac{1-V}{1+V} \quad (4)$$

In equation (2.22) of [3] it was shown that a physical expression for the chiral condensate in a fixed gauge background in the presence of N_f flavors is proportional to

$$\det^{N_f} \left(\frac{1+V}{2} \right) \text{Tr} \frac{1-V}{1+V}, \quad (5)$$

succinctly exhibiting the dichotomy between internal and external propagators. The expression would vanish in finite volumes as long as $N_f > 1$, exactly as expected.

Had g been the internal free fermion propagator, the induced effective action would have had unwanted contributions from ghosts which would survive in the continuum limit, as shown by Pelissetto [4]. But, here the effective action is given by $\det(D)$, not $\det(g^{-1})$, and the free propagator D^{-1} has no ghosts.

In any explicit realization of strictly massless QCD adding a mass terms (see [1,3]) is easy and the dependence on θ_{QCD} can be made explicit. Using the overlap, and making θ space dependent, one immediately obtains a simple lattice chiral Yukawa model.

In [5] the definition of covariant currents was discussed in the general chiral context including the vector-like case. If that definition is applied directly to H one obtains rather ugly expressions

because of nontrivial energy denominators. However, H can always be replaced by an odd monotonic function $f(H)$. In [6] a method to compute D was presented. It is based on a function $F_n(z)$,

$$F_n(z) = \frac{(1+z)^n - (1-z)^n}{(1+z)^n + (1-z)^n} \quad (6)$$

$F_\infty(H) = \epsilon(H)$. A crucial relation is $F_n(F_m(z)) = F_{nm}(z)$. We learn that we can substitute $F_m(H)$ for H . For very large m the energy denominators simplify and nice expressions for the gauge invariant chiral currents can be obtained.

Using the identity

$$F_{2n}(z) = \frac{z}{n} \sum_{s=1}^n \frac{1}{z^2 \cos^2 \frac{\pi}{2n} (s - \frac{1}{2}) + \sin^2 \frac{\pi}{2n} (s - \frac{1}{2})} \quad (7)$$

in conjunction with a shifted CG solver the action of $\epsilon(H)$ on a vector can be easily evaluated [6]. A refinement of this method has appeared in [7].

See [8] for other work related to the above.

Niedermayer, in his plenary talk, applied revisionism to the history of exact chiral symmetry on the lattice. His 46 transparencies are available at <http://pizero.Colorado.EDU/Lattice98/Planary/Niedermayer/> (note unusual spelling). Below are comments I could fit in the remaining allotted space. Some of the comments apply also to the plenary talk given by Sharpe at ICHEP'98.

The renormalization group works by making the ultraviolet complications of field theory emerge in the limit of infinite iteration of a regular step. Regularity means real analyticity in momentum space (locality), real analyticity in field space and real analyticity in the couplings. The iteration is to be applied to a starting point that obeys desired symmetries and the above three-fold analyticity. A lattice-Dirac operator that is analytic in the group valued gauge variables cannot reproduce the disconnected nature of the collection of all smooth continuum gauge fields over a compact manifold. On the lattice any gauge configuration can be smoothly deformed into any other configuration and robust exact zero modes can appear only by going through points where the dependence on the gauge fields is singular. Thus, a lattice model with exact chiral symmetry and an action bilinear in the fermi fields must

violate the condition of analyticity in field space. Therefore, an ideal lattice fixed point action for massless QCD should not be bilinear in the fermi fields. The understanding of the limitations of QCD “perfect actions” (which are not ideal fixed points) is incomplete: to how many loops does “perfection” hold, what is the explicit form of the perfect fermionic action, do any solutions to “perfect fixed point” equations exist, how unique is a solution? Without analyticity in field space there is a danger of loosing universality: for example, we can now obtain arbitrary critical exponents in mean-field. To approach the desired continuum limit the singularities in field space have to decouple from long distance physics. In an e-mail message to Lüscher (Jan. 23, 1998) I suggested that in the vicinity of certain singularities in field space also the locality of the overlap Dirac operator might be lost, but that this would be avoided for actions with a plaquette value restrictively bounded from below (see the 30-th transparency).

The overlap-Dirac operator is *equivalent* to the overlap presented in [1]. Thus, global chiral symmetries are guaranteed, as they would in any theory in which there is no direct coupling between left and right Weyl fermions. These symmetries are masked by the effects of integrating out an infinite number of fermi and pseudo-fermi fields. The simpler $1 + V$ formula appeared first in [2], one year ago, and could not have been prompted by work of Hasenfratz et al. The topological properties were known from [1] and were re-derived in the newer form of [2]. A mass term was introduced in [1] and explicit expressions were written down in [3]. Replacing D by $1 + V$, one can obtain the equations on the 16-th transparency from [3].

For chiral gauge theories [1] one needs the projectors on the positive and negative eigenspaces of H itself, not only H_2 . So long as the gauge fields have zero topology in the sense of the overlap the projectors will be $\mathcal{N} \times \frac{\mathcal{N}}{2}$ matrices representing the action of $\frac{1 \pm \epsilon(H)}{2}$. However, unlike for H_2 , the second factor, $\frac{\mathcal{N}}{2}$, will change when the “instanton number” is non-zero. Explicit formulae for these projectors require the eigenvectors of H and the sign of the related eigenvalues [1]. This is equivalent to spectrally resolving $\epsilon(H)$.

On the 19-th transparency $\epsilon(H) \equiv \gamma_5(1 + V - 1)$ was denoted by $\hat{\gamma}_5$ a notation credited to Hasenfratz, Niedermayer and Lüscher. The eigenvectors and the associated propagator [1] appear on the 44-th transparency. The formulae for topological charge are those of the overlap. As explained in [1] and its predecessor, NPB412(1994)574, the overall phase ambiguity is directly related to anomalies. A fundamental connection between these phases and the algebraic conditions for continuum anomaly cancelation was described in [5]. The importance of the ± 1 modes of V emphasized at the bottom of the 21-st transparency was first stressed in comment (c) of hep-lat/9805027. The introduction of a θ -parameter (topic of the 22-nd transparency) was first discussed in section 9 of [1]. The absence of order a effects is trivial in view of chiral symmetry and was explicitly mentioned in a footnote on page 110 of PLB399(1997). The local properties of the free system were checked in PLB302(1993)62. That the free system is related to GW was mentioned in [3] after eq. (2.24) there. [3] was posted on hep-lat in October 1997. Using the Foerster, Nielsen, Ninomiya mechanism (46-th transparency) was emphasized in [1] and tested in two dimensions thereafter.

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